

# Substitutions for tilings $\{p, q\}$

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In this paper we consider tiling  $\{p, q\}$  of the Euclidean space and of the hyperbolic space, and its dual graph  $\Gamma_{q,p}$  from a combinatorial point of view. A substitution  $\sigma_{q,p}$  on an appropriate finite alphabet is constructed. The homogeneity of graph  $\Gamma_{q,p}$  and its generation function are the basic tools for the construction. The tree associated with substitution  $\sigma_{q,p}$  is a spanning tree of graph  $\Gamma_{q,p}$ . Let  $u_n$  be the number of tiles of tiling  $\{p, q\}$  of generation  $n$ . The characteristic polynomial of the transition matrix of substitution  $\sigma_{q,p}$  is a characteristic polynomial of a linear recurrence. The sequence  $(u_n)_{n \geq 0}$  is a solution of this recurrence. The growth of sequence  $(u_n)_{n \geq 0}$  is given by the dominant root of the characteristic polynomial. The result of this paper is related to [6, 5, 7, 8, 9, 10, 2].

## 1 Combinatorial structure of tiling $\{p, q\}$ induced by the generation function

Here we shall develop [2] using different combinatorial method.

We consider tiling  $\{p, q\}$  of the Euclidean or the hyperbolic plane by regular  $p$ -gons such that each of its vertex is shared by  $q$  gons, i. e.,  $(p-2)(q-2) \geq 4$ , [11], Ch. 5, 3.3. Denote by  $\Gamma_{q,p}$  the dual graph of tiling  $\{p, q\}$ . Remind that the vertices of  $\Gamma_{q,p}$  correspond to the tiles of  $\{p, q\}$ . Two vertices of  $\Gamma_{q,p}$  are connected in this graph iff the corresponding tiles have a common edge. We assume that the dual tiling  $\{q, p\}$  and the dual graph  $\Gamma_{q,p}$  are naturally imbedded in the plane of tiling  $\{p, q\}$ . Precisely, the vertices of  $\Gamma_{q,p}$  are the centres of the corresponding tiles of  $\{p, q\}$ . Let  $V_{q,p}$  be the set of the vertices of graph  $\Gamma_{q,p}$  and let  $E_{q,p}$  be the set of its edges. We denote by  $e_{v_1, v_2}$  the edge of  $\Gamma_{q,p}$ , connecting vertices  $v_1$  and  $v_2$ . Every vertex  $v \in V_{q,p}$  is connected with  $p$  vertices, i.e., the degree of  $v$  in  $\Gamma_{q,p}$  is  $p$ . The vertices, connected with  $v$ , are called neighbors of  $v$ .

We remind the definition of a metric  $d(., .)$  on set  $V_{q,p}$ . Let  $\gamma = e_{v_1, w_2} e_{w_2, w_3} \dots e_{w_k, v_2}$  be a path in graph  $\Gamma_{q,p}$ , connecting vertices  $v_1$  and  $v_2$ . We say that the length  $l(\gamma)$  of path  $\gamma$  is  $l(\gamma) = k$ . Then the metric is defined by  $d(v_1, v_2) = \min\{l(\gamma) : \gamma \text{ is connecting } v_1 \text{ with } v_2\}$ .

Choose and fix a vertex  $\tilde{v} \in V_{q,p}$ . We call this vertex a *root* the graph  $\Gamma_{q,p}$ .

**Definition 1.1** *The generation function  $g : V_{q,p} \longrightarrow \{0, 1, 2, \dots\}$  is defined by  $g(v) = d(\tilde{v}, v)$  for  $v \in V_{q,p}$ . The value  $g(v)$  is called the generation of vertex  $v$ .*

**Remark 1.1** *Every tile of  $\{p, q\}$  is an image of the tile corresponding to the root of  $\Gamma_{q,p}$  by an iteration of symmetries with respect to lines supporting edges of tiles of  $\{p, q\}$ . The minimal number of such symmetries is the generation of the tile. The vertices of graph  $\Gamma_{q,p}$  correspond to the tiles of tiling  $\{p, q\}$  and the generation of a vertex is the generation of the tile, corresponding to it.*

Let  $v'$  be a neighbor of  $v$ . Vertex  $v'$  is called a *successor* of  $v$  if  $g(v') = g(v) + 1$ . In this case  $v$  is called *predecessor* of  $v'$ .

Denote by  $V_n$  the set of all vertices of generation  $n$ , i.e.,  $V_n = \{v \in V_{q,p} : g(v) = n\}$ .

The choice of the root  $\tilde{v}$  and the generation function  $g(n)$  impose a structure on graph  $\Gamma_{q,p}$  and therefore a structure of tiling  $\{p, q\}$ . Our goal is to describe this structure. We shall find a *substitution (a morphism)*  $\sigma_{q,p} : A_{q,p} \longrightarrow A_{q,p}^*$  on an appropriate alphabet  $A_{q,p}$ . This substitution generates set  $V_n$  and the tree associated with  $\sigma_{p,q}$  is a spanning tree of graph  $\Gamma_{q,p}$  with root  $\tilde{v}$ . Moreover, the set of vertices of this tree of generation  $n$  coincides with set  $V_n$ . This implies that sequence  $(u_n)_{n \geq 0}$ ,  $u_n = \text{card}(V_n)$ , is a solution of a linear recurrence with constant coefficients. The characteristic polynomial of this recurrence is the characteristic polynomial of the transition matrix of substitution  $\sigma_{q,p}$ . The homogeneity of graph  $\Gamma_{q,p}$  is very important for the analysis of its combinatorial structure. Moreover, it is known from the proof of Poincaré's theorem that the local considerations are consistent, see [11], Ch. 2, , 1.5. For this reason the proofs of many assertions are by a simple induction on the generation of vertices and, consequently, they are skipped. Let us note that the combinatorial structure of  $\Gamma_{q,p}$  depends on the parity of  $q$ .

### 1.1 Orientation, elementary cycles and relative generation in $\Gamma_{q,p}$

**Lemma 1.1** *There are vertices  $v_1, v_2 \in V_{2k+1,p}$ , connected by an edge  $e_{v_1, v_2} \in E_{2k+1,p}$ , such that  $g(v_1) = g(v_2)$ . There are no such vertices in  $E_{2k,p}$ .*

Proof. The assertion follows by an induction on generation  $g(v_1)$  of vertex  $v_1$ .

**Remark 1.2** *Lemma 1.1 manifests an important difference between graphs  $\Gamma_{2k,p}$  and  $\Gamma_{2k+1,p}$ .*

Let  $e_{v_1, v_2} \in E_{q,p}$  and let  $g(v_2) = g(v_1) + 1$ . We consider edge  $e_{v_1, v_2}$  with the orientation in the direction from  $v_1$  to  $v_2$ .

**Corollary 1.1** *All edges of graph  $\Gamma_{2k,p}$  are oriented, i.e.,  $\Gamma_{2k,p}$  is a directed graph.*

A closed path  $C = e_{v_1, v_2} e_{v_2, v_3} \dots e_{v_k, v_1}$  in  $\Gamma_{q,p}$  is called an *elementary cycle*. All elementary cycles of graph  $\Gamma_{q,p}$  have a length  $q$ , and the elementary cycles are boundaries of the tiles of the dual tiling  $\{q, p\}$  of tiling  $\{p, q\}$

**Definition 1.2** *Let  $g(C) = \min\{g(v_i) : v_i \in V_{q,p}, v_i \in C\}$ .*

We call  $g(C)$  a *generation* of elementary cycle  $C$ .

**Lemma 1.2** *Let  $C$  be an elementary cycle of  $\Gamma_{2k,p}$ . Then the set*

$$\{v_0, v_{1l}, v_{1r}, \dots, v_{(k-1)l}, v_{(k-1)r}, v_k\}$$

*of the vertices of  $C$  satisfies:*

1.  $g(v_0) = g(C)$ ;
2.  $g(v_{il}) = g(v_{ir}) = g(v_0) + i$ ,  $i = 1, 2, \dots, k-1$ ;
3.  $g(v_k) = g(v_0) + k$ .

Proof. By induction on  $g(C)$ .

An elementary cycle of  $\Gamma_{2k,p}$  is represented on Fig. 1(left).

**Lemma 1.3** *Let  $C$  be an elementary cycle of  $\Gamma_{2k+1,p}$ . Two cases are possible:*

1. *The set*

$$\{v_{0l}, v_{0r}, v_{1l}, v_{1r}, \dots, v_{(k-1)l}, v_{(k-1)r}, v_k\}$$

*of the vertices of  $C$  satisfies:  $g(v_{il}) = g(v_{ir}) = g(C) + i$ ,  $i = 0, 1, 2, \dots, k-1$ , and  $g(v_k) = g(C) + k$ ;*

2. *The set*

$$\{v_0, v_{1L}, v_{1R}, \dots, v_{(k-1)L}, v_{(k-1)R}, v_{kL}, v_{kR}\}$$

*of the vertices of  $C$  satisfies:  $g(v_0) = g(C)$ ,  $g(v_{iL}) = g(v_{iR}) = g(C) + i$ ,  $i = 1, 2, \dots, k$ .*

Proof. By induction on  $g(C)$ .

We shall call the elementary cycles of Lemma 1.3.1 *elementary cycles of the first kind*. The elementary cycles of Lemma 1.3.2 are called *elementary cycles of the second kind*. Both of them are represented on Fig. 1(middle, right).

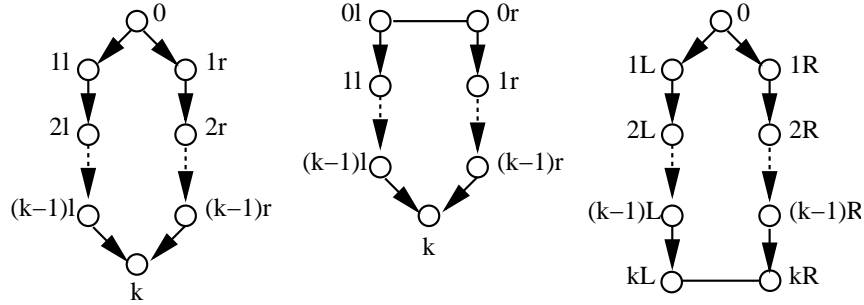


Fig. 1. *Elementary cycle of graph  $\Gamma_{2k,p}$ , Lemma 1.2(left) and elementary cycles of  $\Gamma_{2k+1,p}$ , Lemma 1.3(middle, right).*

**Remark 1.3** *Observe that edges  $e_{v_{0l}, v_{0r}}$  and  $e_{v_{kL}, v_{kR}}$  of the elementary cycles on Fig. 1(middle, right) are not oriented.*

**Definition 1.3** *Let  $C$  be an elementary cycle of  $\Gamma_{q,p}$  and let  $v$  be a vertex of  $C$ . The relative generation  $g_C(v)$  of  $v$  in  $C$  is defined as follows:*

1. Case  $q=2k$ :

$$C = \{v_0, v_{1l}, v_{1r}, \dots, v_{(k-1)l}, v_{(k-1)r}, v_k\},$$

*here we use the notation of Lemma 1.2.*

*Then  $g_C(v_0) = 0$ ,  $g_C(v_{il}) = il$ ,  $g_C(v_{ir}) = ir$  for  $i = 1, \dots, k-1$ , and  $g_C(v_k) = k$ ;*

2. Case  $q=2k+1$ , elementary cycle of the first kind:

$$C = \{v_{0l}, v_{0r}, \dots, v_{(k-1)l}, v_{(k-1)r}, v_k\},$$

here we use the notations of Lemma 1.3(1).

Then  $g_C(v_{il}) = il$ ,  $g_C(v_{ir}) = ir$  for  $i = 0, \dots, k-1$ , and  $g_C(v_k) = k$ ;

3. Case  $q=2k+1$ , elementary cycle of the second type:

$$C = \{v_0, v_{1L}, v_{1R}, \dots, v_{(k-1)L}, v_{(k-1)R}, v_{kL}, v_{kR}\},$$

here we use the notations of Lemma 1.3(2).

Then  $g_C(v_0) = 0$ ,  $g_C(v_{iL}) = iL$ ,  $g_C(v_{iR}) = iR$  for  $i = 1, \dots, k$ .

## 1.2 Types of the vertices of $\Gamma_{q,p}$

Here we shall define a type  $t(v)$  of a vertex  $v \in V_{q,p}$ . For this we consider two cases  $q = 2k$  and  $q = 2k + 1$ .

### Case $q=2k$

Let  $A_{2k,p} = \{0, 1l, 1r, \dots, (k-1)l, (k-1)r, k\}$ . We shall define the map  $t : V_{2k,p} \longrightarrow A_{2k,p}^p$ . For this we need some preliminary considerations.

Let  $v \in V_{2k,p}$  and  $C(v) = \{C_1, \dots, C_p\}$  be the set of all elementary cycles in  $\Gamma_{2k,p}$ , which share  $v$  as a common vertex. We shall define an order of set  $C(v)$ . For this we define the first element of this set and use the positive orientation of the plane (Euclidean or hyperbolic) where tiling  $\{p, 2k\}$  and its dual graph  $\Gamma_{2k,p}$  are embedded.

**Lemma 1.4** *Let  $v \in \Gamma_{2k,p}$ ,  $k \geq 2$ ,  $p \geq 4$  and let  $m(v) = \min\{g(C) : C \in C(v)\}$ . The following cases are possible:*

1. *There is only one elementary cycle  $C_i \in C(v)$  with  $m(v) = g(C_i)$ ;*
2. *There are only two elementary cycles  $C_s, C_t \in C(v)$  with  $m(v) = g(C_s) = g(C_t)$ , and  $g_{C_s}(v) = 1l$ ,  $g_{C_t}(v) = 1r$ ;*
3.  *$g(C_1) = \dots = g(C_p)$  for  $C_i \in C(v)$ ,  $i = 1, \dots, p$ , and  $v = \tilde{v}$ .*

Proof. By induction on  $g(v)$ , see Fig. 2.

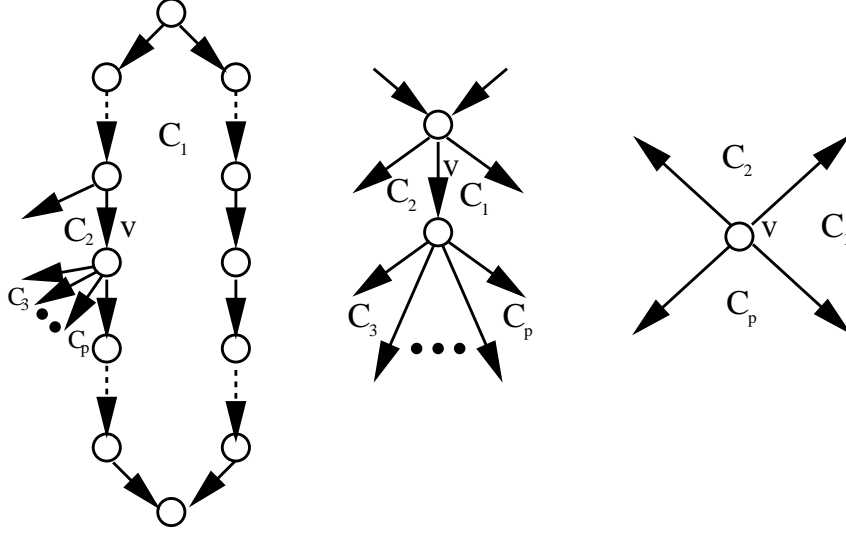


Fig. 2. Elementary cycles of Lemma 1.4(1)(left), (2)(middle), (3)(right).

**Definition 1.4** The first element of set  $C(v)$ ,  $v \in V_{2k,p}$  is defined as follows:

1. It is the unique element  $C_i \in C(v)$  in the case of Lemma 1.4(1);
2. It is element  $C_s \in C(v)$  in the case of Lemma 1.4(2);
3. It is an arbitrary element of  $C(v)$  in the case of Lemma 1.4(3).

**Lemma 1.5** Let  $v \in \Gamma_{2k,3}$ ,  $k \geq 3$  and let  $m(v) = \min\{g(C) : C \in C(v)\}$ . The following cases are possible:

1. There is only one elementary cycle  $C_i \in C(v)$  with  $m(v) = g(C_i)$ ;
2. There are only two elementary cycles  $C_s, C_t \in C(v)$  with  $m(v) = g(C_s) = g(C_t)$ , and one of the following cases hold
  - (a)  $g_{C_s}(v) = 1l$ ,  $g_{C_t}(v) = 1r$ ;
  - (b)  $g_{C_s}(v) = 2l$ ,  $g_{C_t}(v) = 2r$ ;
3.  $g(C_1) = g(C_2) = g(C_3)$  for  $C_i \in C(v)$ ,  $i = 1, 2, 3$ , and  $v = \tilde{v}$ .

Proof. By induction on  $g(v)$  and Lemma 1.3, see Fig. 3.

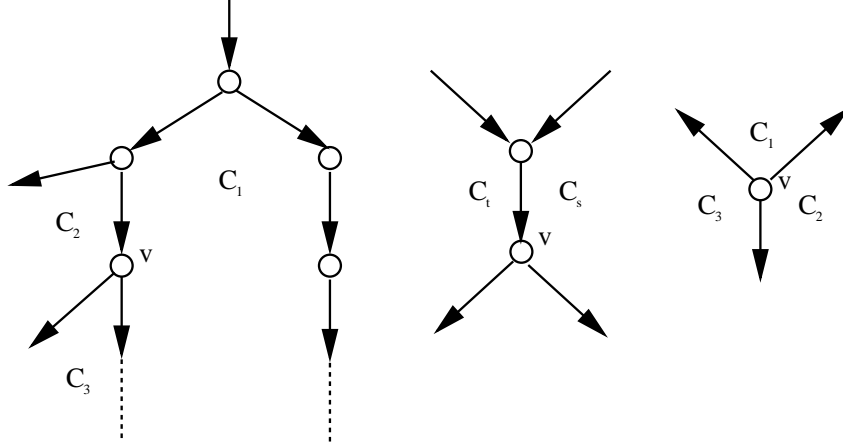


Fig. 3. Elementary cycles of Lemma 1.5(1)(left), (2a)(middle), (3)(right).

**Definition 1.5** The first element of set  $C(v)$ ,  $v \in V_{2k,3}$ ,  $k \geq 3$  is defined as follows:

1. It is the unique element  $C_i \in C(v)$  in the case of Lemma 1.5(1);
2. It is element  $C_s \in C(v)$  in the case of Lemma 1.4(2);
3. It is an arbitrary element of  $C(v)$  in the case of Lemma 1.5(3).

We denote the first element by  $C_1$  in all cases.

The order  $C_1 < \dots < C_p$  of set  $C(v) = \{C_1, \dots, C_p\}$  is defined by the condition that chain  $C_1 \dots C_p$  is in a positive direction in the plane of tiling  $\{p, q\}$ .

**Definition 1.6** Let  $v \in V_{2k,p}$ . The type  $t(v)$  of vertex  $v$  is defined as  $t(v) = g_{C_1}(v) \dots g_{C_p}(v)$ , where  $C(v) = \{C_1, \dots, C_p\}$  and  $C_1 < \dots < C_p$ .

Observe that  $t(v) \in A_{2k,p}^p$ , where  $A_{2k,p} = \{0, 1l, 1r, \dots, (k-1)l, (k-1)r, k\}$ .

**Lemma 1.6** Let  $v \in V_{2k,p}$ ,  $k \geq 2$ ,  $p \geq 4$ . The following types  $t(v)$  are possible:

1.  $a_i = (il)(1r)0^{p-2}$ ,  $i = 1, \dots, k-1$ ;
2.  $a_k = k(1r)0^{p-3}(1l)$ ;
3.  $\bar{a}_j = (jr)0^{p-2}(1l)$ ,  $2 \leq j \leq k-1$ ;
4.  $a_0 = 0^p$ .

Moreover,  $\text{card}\{t(v) : v \in V_{2k,p}\} = 2k-1 = q-1$ .

Proof. By induction on  $g(v)$  and Lemma 1.4, see Fig. 2.

**Lemma 1.7** Let  $v \in V_{2k,3}$ ,  $k \geq 3$ . The following types  $t(v)$  are possible:

1.  $a_i = (il)(1r)0$ ,  $i = 1, \dots, k-1$ ;

2.  $a_k = k(1r)(1l);$
3.  $a_{k+1} = (2l)(2r)0;$
4.  $\bar{a}_j = (jr)0(1l), j = 2, \dots, k-1;$
5.  $a_0 = 0^3.$

Proof. By induction on  $g(v)$ . See Fig. 3.

**Case  $q=2k+1$**

Let

$$A_{2k+1,p} = \{0l, 0r, \dots, (k-1)l, (k-1)r, k, 0, 1L, 1R, \dots, kL, kR\}.$$

We shall define a map  $t : V_{2k+1,p} \longrightarrow A_{2k+1,p}^p$ . For this we follow the procedure used in the previous case  $q = 2k$ .

Let  $C(v) = \{C_1, \dots, C_p\}$ , where  $v$  is a vertex with  $v \in C_i, i = 1, \dots, p$ . We shall define an order  $C_1 < \dots < C_p$  of set  $C(v)$ . Then  $t(v)$  is defined as  $t(v) = g_{C_1}(v) \dots g_{C_p}(v)$ . To define the order of  $C(v)$  we define its first element and use the positive orientation of the plane of tiling  $\{p, 2k+1\}$ .

**Lemma 1.8** *Let  $v \in V_{2k+1,p}, k \geq 2, p \geq 4$  and let  $m(v) = \min\{g(C) : C \in C(v)\}$ . The following cases are possible:*

1. *There is only one elementary cycle  $C_i \in C(v)$  with  $m(v) = g(C_i)$ ;*
2. *There are only two elementary cycles  $C_s, C_t \in C(v)$  with  $m(v) = g(C_s) = g(C_t)$ , and one of the following cases hold:*
  - (a)  $g_{C_s}(v) = 1l, g_{C_t}(v) = 1R;$
  - (b)  $g_{C_s}(v) = 1r, g_{C_t}(v) = 1L;$
  - (c)  $g_{C_s}(v) = 1L, g_{C_t}(v) = 1R;$
3.  $g(C_1) = \dots = g(C_p)$  for  $C_i \in C(v), i = 1, \dots, p$ .

Proof. By induction on  $g(v)$  and Lemma 1.3, see Fig. 4a and Fig. 4b.

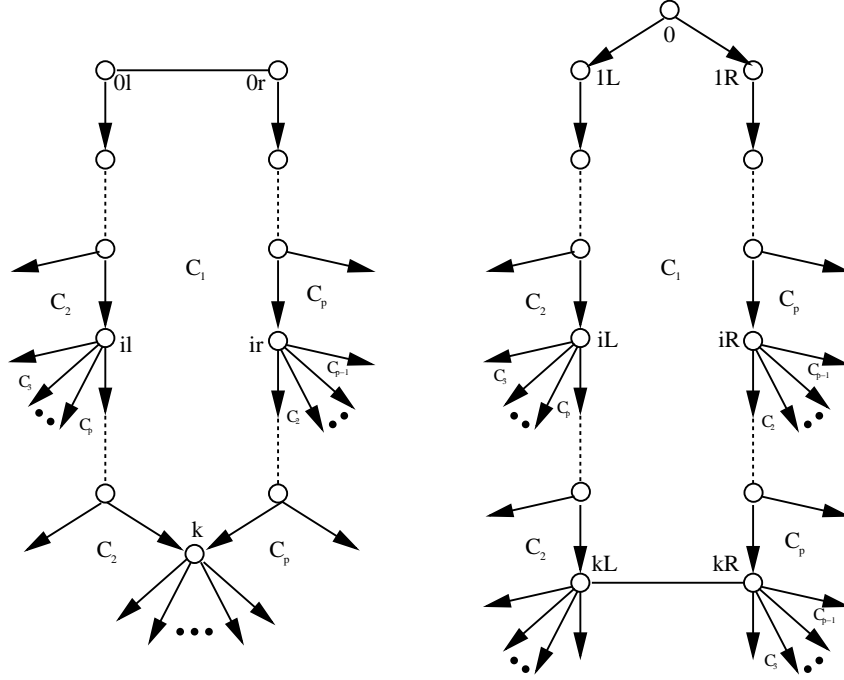


Fig. 4a. Elementary cycles, Lemma 1.8(1).

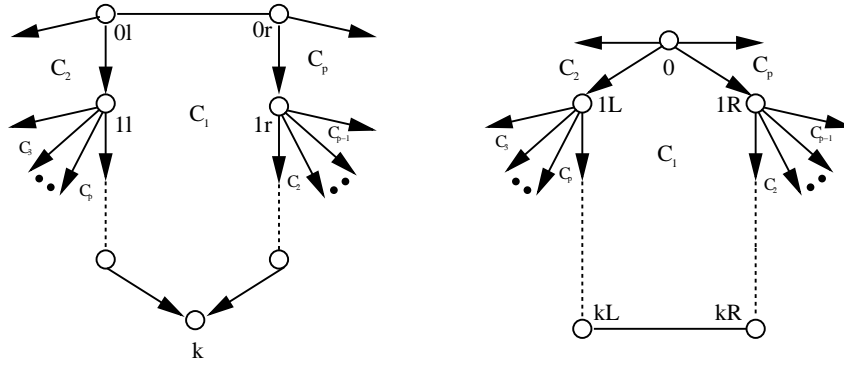


Fig. 4b. Elementary cycles, Lemma 1.8(2).

**Definition 1.7** The first element of set  $C(v)$  is defined as follows:

1.  $C_i$  in the case of Lemma 1.8(1);
2.  $C_s$  in the case of Lemma 1.8(2);
3. It is an arbitrary element of  $C(v)$  in the case of Lemma 1.8(3).

**Lemma 1.9** Let  $v \in V_{2k+1,3}$ ,  $k \geq 3$  and let  $m(v) = \min\{g(C) : C \in C(v)\}$ . The following cases are possible:



1. There is only one elementary cycle  $C_i \in C(v)$  with  $m(v) = g(C_i)$ ;
2. There are only two elementary cycles  $C_s, C_t \in C(v)$  with  $m(v) = g(C_s) = g(C_t)$ , and one of the following cases hold:
  - (a)  $g_{C_s}(v) = 1l, g_{C_t}(v) = 1R$  ;
  - (b)  $g_{C_s}(v) = 1r, g_{C_t}(v) = 1L$  ;
  - (c)  $g_{C_s}(v) = 1L, g_{C_t}(v) = 1R$ ;
  - (d)  $g_{C_s}(v) = 2L, g_{C_t}(v) = 2R$ ;
3.  $g(C_1) = \dots = g(C_p)$  for  $C_i \in C(v)$ ,  $i = 1, \dots, p$ .

Proof. By induction on  $g(v)$  and Lemma 1.3 see Fig. 5.

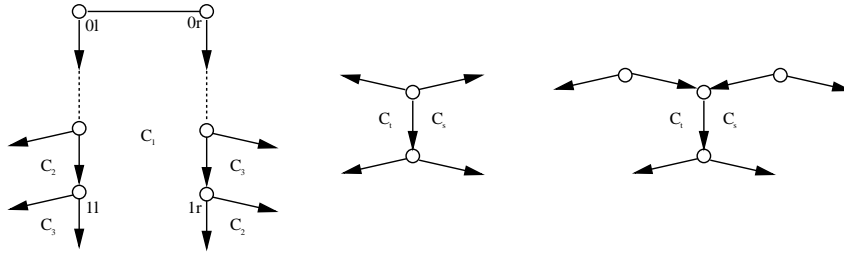


Fig. 5. Elementary cycles, Lemma 1.9(1)(left), Lemma 1.9(2a)(middle), Lemma 1.9(2b)(right).

**Definition 1.8** The first element of set  $C(v)$  is defined as follows:

1.  $C_i$  in the case of Lemma 1.9(1);
2.  $C_s$  in the case of Lemma 1.9(2);
3. It is an arbitrary element of  $C(v)$  in the case of Lemma 1.9(3).

**Lemma 1.10** Let  $v \in V_{3,p}$ ,  $p \geq 6$  and let  $m(v) = \min\{g(C) : C \in C(v)\}$ . The following cases are possible:

1. There are only two elementary cycles  $C_s, C_t \in C(v)$  with  $m(v) = g(C_s) = g(C_t)$ , and  $g_{C_s}(v) = 1L, g_{C_t}(v) = 1R$  ;
2. There are 3 elementary cycles  $C_s, C_t, C_u \in C(v)$  with  $m(v) = g(C_s) = g(C_t) = g(C_u)$ , and  $g_{C_s}(v) = 1L, g_{C_t}(v) = 1, g_{C_u}(v) = 1R$  ;
3.  $g(C_1) = \dots = g(C_p)$  for  $C_i \in C(v)$ ,  $i = 1, \dots, p$ .

Proof. By induction on  $g(v)$ , see Fig. 6.

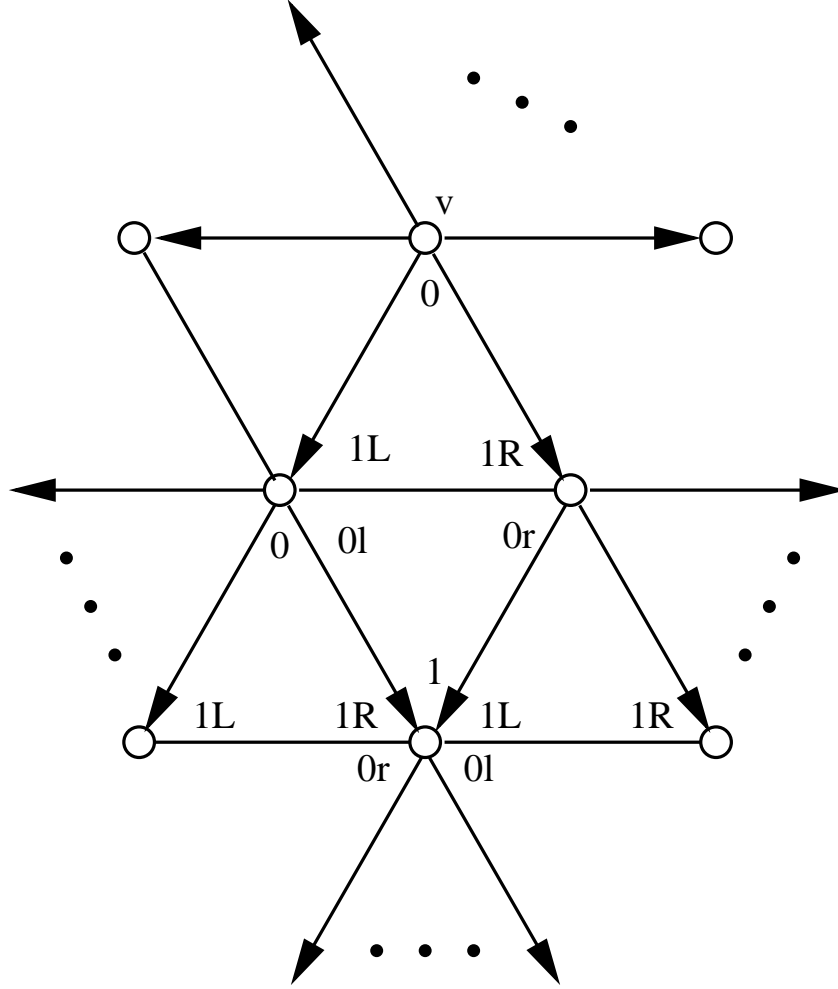


Fig. 6. Elementary cycles, Lemma 1.10.

**Definition 1.9** *The first element of set  $C(v)$  is defined as follows:*

1.  $C_i$  in the case of Lemma 1.10(1);
2.  $C_s$  in the case of Lemma 1.10(2);
3. It is an arbitrary element of  $C(v)$  in the case of Lemma 1.10(3).

We denote the first element of  $C(v)$  by  $C_1$  in all this cases. The order  $C_1 < \dots < C_p$  of  $C(v) = \{C_1, \dots, C_p\}$  is defined by the condition that chain  $C_1 \dots C_p$  is in the positive direction in the plane of tiling  $\{p, 2k+1\}$ .

**Lemma 1.11** *Let  $v \in V_{2k+1,p}$ ,  $k \geq 2$ ,  $p \geq 4$ . The following types  $t(v)$  are possible:*

1.  $a_i = (il)(1R)0^{p-2}$ ,  $i = 1, \dots, k-1$ ;
2.  $a_k = k(1R)0^{p-3}(1L)$ ;
3.  $b_i = (iL)(1R)0^{p-2}$ ,  $i = 1, \dots, k-1$ ;

4.  $b_k = (kL)(1R)0^{p-3}(0l);$
5.  $\bar{a}_j = (jr)0^{p-2}(1L), \ 1 \leq j \leq k-1;$
6.  $\bar{b}_j = (jR)0^{p-2}(1L), \ 2 \leq j \leq k-1;$
7.  $\bar{b}_k = (kR)(0r)0^{p-3}(1L);$
8.  $a_0 = 0^p.$

Proof. Induction on  $g(v)$  and Lemma 1.8, see Fig. 4.

**Lemma 1.12** *Let  $v \in V_{2k+1,3}$ ,  $k \geq 3$ . The following types  $t(v)$  are possible:*

1.  $a_1 = (2R)0(1l);$
2.  $a_i = (il)(1R)0, \ i = 2, \dots, k-1;$
3.  $a_k = k(1R)(1L);$
4.  $a_{k+1} = (2R)(2L)0$
5.  $b_i = (iL)(1R)0, \ i = 1, \dots, k-1;$
6.  $b_k = (kL)(1R)(0l);$
7.  $\bar{a}_1 = (2L)(1r)0;$
8.  $\bar{a}_j = (jr)0(1L), \ 2 \leq j \leq k-1;$
9.  $\bar{b}_j = (jR)0(1L), \ 2 \leq j \leq k-1;$
10.  $\bar{b}_k = (kR)(0r)(1L);$
11.  $a_0 = 0^3.$

Moreover,  $\text{card}\{t(v) : v \in V_{2k+1,p}, \ k > 1, \ p \geq 4\} = 4k - 1 = 2q - 3.$

Proof. By induction on  $g(v)$  and Lemma 1.9, see Fig. 5.

**Remark 1.4** *For the sake of simplicity of the notations we take  $\bar{a}_1 = (1r)0^{p-2}(1L)$  instead of  $(1L)(1r)0^{p-2}.$*

**Lemma 1.13** *Let  $v \in V_{3,p}$ ,  $k = 1$ ,  $p \geq 6$ . The following types  $t(v)$  are possible:*

1.  $a = (1L)(1R)(0r)0^{p-4}(0l);$
2.  $b = (1L)1(1R)(0r)0^{p-5}(0l).$
3.  $a_0 = 0^p.$

Proof. By induction on  $g(v)$  and Lemma 1.10, see Fig. 6.

### 1.3 Types of the successors of the vertices

**Lemma 1.14** *Let  $v \in V_{2k,p}$ ,  $k \geq 2, p \geq 4$ . Let  $v_1, \dots, v_l$  be the successors of vertex  $v$ . The types of  $v_1, \dots, v_l$  are given by  $t(v) \rightarrow t(v_1) \dots t(v_l)$ :*

$$1. t(v) = a_i \rightarrow \bar{a}_2(a_1)^{p-3}a_{i+1}, \quad 1 \leq i \leq k-1;$$

$$2. t(v) = a_k \rightarrow \bar{a}_2(a_1)^{p-4}a_2;$$

$$3. t(v) = \bar{a}_j \rightarrow \bar{a}_{j+1}(a_1)^{p-3}a_2, \quad 2 \leq j \leq k-2;$$

$$4. t(v) = \bar{a}_{k-1} \rightarrow a_k(a_1)^{p-3}a_2;$$

$$5. t(v) = a_0 \rightarrow (a_1)^p.$$

Proof. The assertion follows from Lemma 1.6, use Fig. 2.

**Lemma 1.15** *Let  $v \in V_{2k,3}$ ,  $k \geq 3$ . Let  $v_1, \dots, v_l$  be the successors of vertex  $v$ . The types of  $v_1, \dots, v_l$  are given by  $t(v) \rightarrow t(v_1) \dots t(v_k)$ :*

$$1. t(v) = a_i \rightarrow \bar{a}_2a_{i+1}, \quad 1 \leq i \leq k-1;$$

$$2. t(v) = a_k \rightarrow a_{k+1};$$

$$3. t(v) = a_{k+1} \rightarrow \bar{a}_3a_3;$$

$$4. t(v) = \bar{a}_j \rightarrow \bar{a}_{j+1}a_2, \quad 2 \leq j \leq k-1;$$

$$5. t(v) = a_0 \rightarrow (a_1)^p.$$

Proof. The assertion follows from Lemma 1.7, use Fig. 3.

**Lemma 1.16** *Let  $v \in V_{2k+1,p}$ ,  $k \geq 2, p \geq 4$ . Let  $v_1, \dots, v_l$  be the successors of vertex  $v$ . The types of  $v_1, \dots, v_l$  are given by  $t(v) \rightarrow t(v_1) \dots t(v_l)$ :*

$$1. t(v) = a_i \rightarrow \bar{b}_2(b_1)^{p-3}a_{i+1}, \quad 1 \leq i \leq k-1;$$

$$2. t(v) = a_k \rightarrow \bar{b}_2(b_1)^{p-4}b_2;$$

$$3. t(v) = \bar{a}_i \rightarrow \bar{a}_{i+1}(b_1)^{p-3}b_2, \quad 1 \leq j \leq k-2;$$

$$4. t(v) = \bar{a}_{k-1} \rightarrow a_k(b_1)^{p-3}b_2;$$

$$5. t(v) = b_i \rightarrow \bar{b}_2(b_1)^{p-3}b_{i+1}, \quad 1 \leq i \leq k-1;$$

$$6. t(v) = b_k \rightarrow \bar{b}_2(b_1)^{p-4}a_1;$$

$$7. t(v) = \bar{b}_j \rightarrow \bar{b}_{i+1}(b_1)^{p-3}b_2, \quad 2 \leq j \leq k-2;$$

$$8. t(v) = \bar{b}_{k-1} \rightarrow b_k(b_1)^{p-3}b_2;$$

$$9. t(v) = \bar{b}_k \rightarrow \bar{a}_1(b_1)^{p-4}b_2;$$

$$10. t(v) = a_0 \rightarrow (b_1)^p.$$

Proof. The assertion follows from Lemma 1.11, use Fig. 4a, 4b.

**Remark 1.5** Observe that  $b_k$  and  $\bar{b}_k$  are connected in  $\Gamma_{q,p}$  by a non oriented edge.

**Lemma 1.17** Let  $v \in V_{2k+1,3}$ ,  $k \geq 3$ . Let  $v_1, \dots, v_l$  be the successors of vertex  $v$ . The types of  $v_1, \dots, v_l$  are given by  $t(v) \rightarrow t(v_1) \dots t(v_l)$ :

$$1. t(v) = a_1 \rightarrow \bar{b}_3a_2;$$

$$2. t(v) = a_i \rightarrow \bar{b}_2a_{i+1}, \quad i = 2, \dots, k-1;$$

$$3. t(v) = a_k \rightarrow a_{k+1};$$

$$4. t(v) = a_{k+1} \rightarrow \bar{b}_3b_3;$$

$$5. t(v) = \bar{a}_1 \rightarrow \bar{a}_2b_3;$$

$$6. t(v) = \bar{a}_i \rightarrow \bar{a}_{i+1}b_2, \quad i = 2, \dots, k-1;$$

$$7. t(v) = b_i \rightarrow \bar{b}_2b_{i+1}, \quad 1 \leq i \leq k-1;$$

$$8. t(v) = b_k \rightarrow a_1;$$

$$9. t(v) = \bar{b}_i \rightarrow \bar{b}_{i+1}b_2, \quad i = 2, \dots, k-1;$$

$$10. t(v) = \bar{b}_k \rightarrow \bar{a}_1;$$

11.  $t(v) = a_0 \rightarrow (b_1)^p$ .

Proof. The assertion follows from Lemma 1.12 and Lemma 1.9, use Fig. 5.

**Lemma 1.18** *Let  $v \in V_{3,p}, p \geq 6$ . Let  $v_1, \dots, v_l$  be the successors of vertex  $v$ . The types of  $v_1, \dots, v_l$  are given by  $t(v) \rightarrow t(v_1) \dots t(v_l)$ :*

1.  $t(v) = a \rightarrow ba^{p-5}b$ ;

2.  $t(v) = b \rightarrow ba^{p-6}b$ ;

3.  $t(v) = a_0 \rightarrow (b)^p$ .

Proof. The assertion follows from Lemma 1.13, use Fig. 6.

#### 1.4 Substitution generating a spanning tree of $\Gamma_{q,p}$

Let  $\Omega_{q,p} = \{t(v) : v \in V_{q,p}\}$ . Here we shall define a substitution (morphism)  $\sigma_{q,p} : \Omega_{q,p} \longrightarrow \Omega_{q,p}^*$ .

The  $n$ -th iteration  $\sigma_{q,p}^n(0^p)$  of  $0^p \in \Omega_{q,p}$  is a concatenation of words belonging to the set  $\{\sigma_{q,p}(w) : w \in \Omega_{q,p}\}$ .

The tree  $T_{q,p}$  associated with the orbit  $(\sigma_{q,p}^n(0^p))_{n \geq 0}$  of word  $0^p \in \Omega_{q,p}$  is a spanning tree of graph  $\Gamma_{q,p}$ . Remind that tree  $T_{q,p}$  is defined as follows. Its vertices are the points of orbit  $(\sigma_{q,p}^n(0^p))_{n \geq 0}$ . The root of  $T_{q,p}$  is  $0^p$ . Let  $\omega \in T_{q,p}$ , then the successors of  $\omega$  in tree  $T_{q,p}$  are all words belonging to  $\sigma_{q,p}(\omega)$ .

The embedding of tree  $T_{q,p}$  in graph  $\Gamma_{q,p}$  is defined by induction on the generation of the vertices. The root  $0^p$  of  $T_{q,p}$  is identified with the root  $\tilde{v}$  of  $\Gamma_{q,p}$ . Assume that the embedding is defined for all vertices of  $T_{q,p}$  of generation  $\leq k$ , and all edges connecting them. Let  $\omega \in T_{q,p}, v \in \Gamma_{q,p}$  and let  $\omega, v$  be of the same generation  $k$ . Identify the words in  $\sigma_{q,p}(\omega)$  with the successors of  $v$  of the same type in the direction of the positive orientation of the plane.

We consider two cases:  $q = 2k$  and  $q = 2k + 1$  for the definition of substitution  $\sigma_{q,p}$ .

##### Case $q = 2k, k \geq 3, p \geq 4$

In this case

$$\Omega_{2k,p} = A_{2k,p}^p = \{a_0, a_1, \dots, a_k, \bar{a}_2, \bar{a}_3, \dots, \bar{a}_{k-1}\}.$$

Here we use the notations of Lemma 1.6 and Lemma 1.14.

**Definition 1.10** *Let  $q = 2k, k \geq 3, p \geq 4$ . The substitution  $\sigma_{2k,p} : \Omega_{2k,p} \longrightarrow \Omega_{2k,p}^*$  is defined as*

follows.

$$\begin{aligned}
\sigma_{2k,p}(a_0) &= (a_1)^p; \\
\sigma_{2k,p}(a_i) &= \bar{a}_2(a_1)^{p-3}a_{i+1}, \quad 1 \leq i \leq k-2; \\
\sigma_{2k,p}(a_{k-1}) &= \bar{a}_2(a_1)^{p-3}; \\
\sigma_{2k,p}(a_k) &= \bar{a}_2(a_1)^{p-4}a_2; \\
\sigma_{2k,p}(\bar{a}_i) &= \bar{a}_{j+1}(a_1)^{p-3}a_2, \quad 2 \leq i \leq k-2; \\
\sigma_{2k,p}(\bar{a}_{k-1}) &= a_k(a_1)^{p-3}a_2.
\end{aligned}$$

**Case  $q = 4, p = 4$**

In this case

$$\Omega_{4,4} = A_{4,4}^4 = \{a_0, a_1, a_2\}.$$

**Definition 1.11** Let  $q = 4, p = 4$ . The substitution  $\sigma_{4,4} : \Omega_{4,4} \longrightarrow \Omega_{4,4}^*$  is defined as follows.

$$\begin{aligned}
\sigma_{4,4}(a_0) &= (a_1)^4; \\
\sigma_{4,4}(a_1) &= a_2a_1; \\
\sigma_{4,4}(a_2) &= a_2.
\end{aligned}$$

**Case,  $q = 2k, k \geq 4, p = 3$**

In this case

$$\Omega_{2k,3} = A_{2k,3}^3 = \{a_0, a_1, \dots, a_{k+1}, \bar{a}_2, \bar{a}_3\}.$$

**Definition 1.12** Let  $q = 2k, k \geq 4, p = 3$ . The substitution  $\sigma_{2k,3} : \Omega_{2k,3} \longrightarrow \Omega_{2k,3}^*$  is defined as follows.

$$\begin{aligned}
\sigma_{2k,3}(a_0) &= (a_1)^3; \\
\sigma_{2k,3}(a_i) &= \bar{a}_2a_{i+1}, \quad 1 \leq i \leq k-2; \\
\sigma_{2k,3}(a_{k-1}) &= \bar{a}_2; \\
\sigma_{2k,3}(a_k) &= a_{k+1}; \\
\sigma_{2k,3}(a_{k+1}) &= \bar{a}_3a_3; \\
\sigma_{2k,3}(\bar{a}_j) &= \bar{a}_{j+1}a_2, \quad 2 \leq j \leq k-2; \\
\sigma_{2k,3}(\bar{a}_{k-1}) &= a_k a_2.
\end{aligned}$$

Case  $q = 6, k = 3, p = 3$

In this case

$$\Omega_{6,3} = A_{6,3}^3 = \{a_0, a_1, a_2, a_3, a_4, \bar{a}_2\}.$$

**Definition 1.13** *Let  $q = 6, k = 3, p = 3$ . The substitution  $\sigma_{6,3} : \Omega_{6,3} \longrightarrow \Omega_{6,3}^*$  is defined as follows:*

$$\sigma_{6,3}(a_0) = (a_1)^3;$$

$$\sigma_{6,3}(a_1) = \bar{a}_2 a_2;$$

$$\sigma_{6,3}(a_2) = \bar{a}_2;$$

$$\sigma_{6,3}(a_3) = a_4;$$

$$\sigma_{6,3}(a_4) = a_3;$$

$$\sigma_{6,3}(\bar{a}_2) = a_3 a_2.$$

Case  $q = 2k + 1, k \geq 2, p \geq 4$

In this case

$$\Omega_{2k+1,p} = A_{2k+1,p}^p = \{a_0, a_1, \dots, a_k, \bar{a}_1, \bar{a}_2, \dots, \bar{a}_{k-1}, b_1, \dots, b_k, \bar{b}_2, \bar{b}_3, \dots, \bar{b}_k\}.$$

Here we use the notations of Lemmas 1.11 and 1.16.

**Definition 1.14** *Let  $q = 2k + 1, k \geq 2, p \geq 4$ . The substitution  $\sigma_{2k+1,p} : \Omega_{2k+1,p} \longrightarrow \Omega_{2k+1,p}^*$  is defined as follows.*

$$\sigma_{2k+1,p}(a_0) = (b_1)^p;$$

$$\sigma_{2k+1,p}(a_i) = \bar{b}_2 (b_1)^{p-3} a_{i+1}, \quad 1 \leq i \leq k-1;$$

$$\sigma_{2k+1,p}(a_k) = \bar{b}_2 (b_1)^{p-4} b_2;$$

$$\sigma_{2k+1,p}(b_i) = \bar{b}_2 (b_1)^{p-3} b_{i+1}, \quad 1 \leq i \leq k-1;$$

$$\sigma_{2k+1,p}(b_k) = \bar{b}_2 (b_1)^{p-4} a_1;$$

$$\sigma_{2k+1,p}(\bar{a}_j) = \bar{a}_{j+1} (b_1)^{p-3} b_2, \quad 1 \leq j \leq k-2;$$

$$\sigma_{2k+1,p}(\bar{a}_{k-1}) = (b_1)^{p-3} b_2;$$

$$\sigma_{2k+1,p}(\bar{b}_j) = \bar{b}_{j+1} (b_1)^{p-3} b_2, \quad 2 \leq j \leq k-1;$$

$$\sigma_{2k+1,p}(\bar{b}_k) = \bar{a}_1 (b_1)^{p-4} b_2.$$



Case  $q = 2k + 1, k \geq 3, p = 3$

In this case

$$\Omega_{2k+1,3} = A_{2k+1,3}^3 = \{a_0, a_1, \dots, a_k, a_{k+1}, \bar{a}_1, \bar{a}_2, \dots, \bar{a}_{k-1}, b_1, \dots, b_k, \bar{b}_2, \bar{b}_3, \dots, \bar{b}_k\}.$$

Here we use the notations of Lemma 1.11.

**Definition 1.15** *Let  $q = 2k + 1, k \geq 3, p = 3$ . The substitution  $\sigma_{2k+1,3} : \Omega_{2k+1,3} \longrightarrow \Omega_{2k+1,3}^*$  is defined as follows.*

$$\begin{aligned} \sigma_{2k+1,3}(a_0) &= (b_1)^3; \\ \sigma_{2k+1,3}(a_1) &= \bar{b}_3 a_2 \\ \sigma_{2k+1,3}(a_i) &= \bar{b}_2 a_{i+1}, \quad 2 \leq i \leq k-1; \\ \sigma_{2k+1,3}(a_k) &= a_{k+1}; \\ \sigma_{2k+1,3}(a_{k+1}) &= \bar{b}_3 b_3; \\ \sigma_{2k+1,3}(b_i) &= \bar{b}_2 b_{i+1}, \quad 1 \leq i \leq k-1; \\ \sigma_{2k+1,3}(b_k) &= a_1; \\ \sigma_{2k+1,3}(\bar{a}_1) &= \bar{a}_2 b_3; \\ \sigma_{2k+1,3}(\bar{a}_j) &= \bar{a}_{j+1} b_2, \quad 2 \leq j \leq k-2; \\ \sigma_{2k+1,3}(\bar{a}_{k-1}) &= b_2; \\ \sigma_{2k+1,3}(\bar{b}_j) &= \bar{b}_{j+1} b_2, \quad 2 \leq j \leq k-1; \\ \sigma_{2k+1,3}(\bar{b}_k) &= \bar{a}_1. \end{aligned}$$

Case  $q = 3, p \geq 6$

In this case

$$\Omega_{3,p} = A_{2k+1,p}^p = \{a_0, a, b\}.$$

We use the notations of Lemma 1.18

**Definition 1.16** *Let  $v \in V_{3,p}, p \geq 6$ . The substitution  $\sigma_{3,p} : \Omega_{3,p} \longrightarrow \Omega_{3,p}^*$  is defined as follows.*

$$\begin{aligned} \sigma_{3,p}(a_0) &= b^p; \\ \sigma_{3,p}(a) &= a^{p-5} b; \\ \sigma_{3,p}(b) &= a^{p-6} b. \end{aligned}$$

**Theorem 1.1** *The tree  $T_{q,p}$  corresponding to the orbit  $(\sigma_{q,p}^n(a_0))_{n \geq 0}$  of  $a_0$  with respect to the substitution  $\sigma_{q,p} : \Omega_{q,p} \longrightarrow \Omega_{q,p}^*$  is a spanning tree of graph  $\Gamma_{q,p}$ . The vertices of  $\Gamma_{q,p}$  of generation  $n$  correspond to the letters of word  $\sigma_{q,p}^n(a_0) \in \Omega_{q,p}$ .*

Proof. Follows from the construction.

### 1.5 The transition graph, the transition matrix of substitution $\tilde{\sigma}_{q,p}$

Here we consider the substitution  $\tilde{\sigma}_{q,p} = \sigma_{q,p} \mid \tilde{\Omega}_{q,p} : \tilde{\Omega}_{q,p} \longrightarrow \tilde{\Omega}_{q,p}^*$ , where  $\tilde{\Omega}_{q,p} = \Omega_{q,p} \setminus \{a_0\}$ .

Denote the transition(incidence) matrix of substitution  $\tilde{\sigma}_{q,p}$  by  $M_{q,p}$ . The rows and the columns of  $M_{q,p}$  are labeled by the elements  $a_1, a_2, \dots, b_1, b_2, \dots, \bar{a}_1, \bar{a}_2, \dots, \bar{b}_2, \bar{b}_3, \dots$  of  $\tilde{\Omega}_{q,p}$  in this order. The elements  $m_{\alpha,\beta} \in M_{q,p}$ ,  $\alpha, \beta \in \tilde{\Omega}_{q,p}$  are defined as

$$m_{\alpha,\beta} = \text{card}\{\beta : \beta \text{ is a letter of word } \tilde{\sigma}_{q,p}(\alpha)\}.$$

We denote by  $\Gamma(\tilde{\sigma}_{q,p})$  the transition graph of substitution  $\tilde{\sigma}_{q,p} : \tilde{\Omega}_{q,p} \longrightarrow \tilde{\Omega}_{q,p}^*$ . Graph  $\Gamma(\tilde{\sigma}_{q,p})$  is a directed graph with transition matrix  $M_{q,p}$ , [4], ch. 15, [1], ch. 3. A directed graph is called strongly connected if, for each pair of vertices  $v_1$  and  $v_2$ , there is a directed path connecting  $v_1$  to  $v_2$ .

Remind that the index of imprimitivity  $h_{q,p}$  of the strongly connected graph  $\Gamma(\tilde{\sigma}_{q,p})$  is the greatest common divisor of the lengths of the closed directed paths on  $\Gamma(\tilde{\sigma}_{q,p})$ , [1], ch. 3, 3.5, [4], ch. 15, 15.6.

**Lemma 1.19** *The transition graph  $\Gamma(\tilde{\sigma}_{q,p})$  has the following properties:*

1.  $\Gamma(\tilde{\sigma}_{q,p})$  is strongly connected for  $q = 2k, k \geq 3, p \geq 4$ . The index of imprimitivity  $h_{2k,p}$  of  $\Gamma(\tilde{\sigma}_{q,p})$  is one;
2.  $\Gamma(\tilde{\sigma}_{4,4})$  has two strongly connected components  $\{a_1\}$  and  $\{a_2\}$ ;
3.  $\Gamma(\tilde{\sigma}_{2k,3})$ ,  $k \geq 4$  has one strongly connected components. It contains all vertices different from  $a_1$ . The vertex  $a_1$  is isolated. The index of imprimitivity  $h_{2k,3}$  of the strongly connected component is one;
4.  $\Gamma(\tilde{\sigma}_{6,3})$  has two strongly connected components  $\{a_2, \bar{a}_2\}$ ,  $\{a_3, a_4\}$ . The vertex  $a_1$  is isolated;
5.  $\Gamma(\tilde{\sigma}_{2k+1,p})$ ,  $k \geq 2, p \geq 4$  is strongly connected. The index of imprimitivity  $h_{2k+1,p}$  of  $\Gamma(\tilde{\sigma}_{2k+1,p})$  is one;
6.  $\Gamma(\tilde{\sigma}_{2k+1,3})$ ,  $k \geq 3$  has one strongly connected component, containing all vertices different from  $b_1$ . The index of imprimitivity  $h_{2k+1,3}$  of this component is one;
7.  $\Gamma(\tilde{\sigma}_{3,p})$ ,  $p \geq 7$  is strongly connected. The index of imprimitivity  $h_{3,p}$  of  $\Gamma(\tilde{\sigma}_{3,p})$  is one;
8.  $\Gamma(\tilde{\sigma}_{3,6})$  has two strongly connected components  $\{a\}$  and  $\{b\}$ .

Remind that the transition matrix of a strongly connected graph is irreducible, [4], ch. 15. Let  $M$  be a nonnegative irreducible matrix, then the Perron-Frobenius theorem gives.:

1. Matrix  $M$  has a positive eigenvalue,  $r$ , equal to the spectral radius of  $M$ ;
2. There is a strongly positive (right) eigenvector associated with eigenvalue  $r$ ;

3. Eigenvalue  $r$  is a simple root of the characteristic polynomial of  $M$ . If  $M = (m_{i,j})$  and  $\sigma_j = \sum_k m_{j,k}$ , then  $\min_j \sigma_j \leq r \leq \max_j \sigma_j$ ;
4. Let the index of imprimitivity  $h = h(M) > 1$ . Then matrix  $M$  has eigenvalues  $\lambda_0 = r, \lambda_2, \dots, \lambda_{h-1}$  with  $\lambda_s = \exp \frac{2s\pi}{h} r, 0 \leq s \leq h-1$ . All these eigenvalues are simple;
5. The index of imprimitivity  $h = h(M)$  of matrix  $M$  is equal to the index of imprimitivity of the transition graph of matrix  $M$ ;
6. Matrix  $M$  is called primitive if  $h(M) = 1$ . A primitive matrix  $M$  has a simple positive eigenvalue  $r$ , which is dominant, i.e.,  $r > \lambda$  for every eigenvalue  $\lambda \neq r$  of  $M$ ,

see [4], Ch. 15; [1], ch. 3.

**Corollary 1.2** *The transition matrix  $M_{p,q}$  of substitution  $\tilde{\sigma}_{q,p}$  has the following properties.*

1.  $M_{2k,p}$ ,  $k \geq 3$ ,  $p \geq 4$  is primitive and it has a dominant single positive eigenvalue  $r_{2k,p} > 1$ ;
2.  $M_{4,4}$  is reducible;
3.  $M_{2k,3}$ ,  $k \geq 4$  is reducible, but it has a dominant single positive eigenvalue  $r_{2k,3} > 1$ ;
4.  $M_{6,3}$  is reducible;
5.  $M_{2k+1,p}$ ,  $k \geq 2$ ,  $p \geq 4$  is primitive and it has a dominant single positive eigenvalue  $r_{2k+1,p} > 1$ ;
6.  $M_{2k+1,3}$ ,  $k \geq 3$  is matrix and it has a dominant single positive eigenvalue  $r_{2k+1,3} > 1$ ;
7.  $M_{3,p}$ ,  $p \geq 7$  is primitive and it has a dominant single positive eigenvalue  $r_{3,p} > 1$ ;
8.  $M_{3,6}$  is reducible.

## 1.6 Characteristic polynomial of substitution $\tilde{\sigma}_{q,p}$ and number of elements of generation $n$ of $\Gamma_{q,p}$

By a definition, the characteristic polynomial  $\chi_{q,p}(x)$  of substitution  $\tilde{\sigma}_{q,p}$  is the characteristic polynomial of the matrix  $M_{q,p}$ . Let

$$\chi_{q,p}(x) = x^N - c_1 x^{N-1} - \dots - c_{N-l-1} x - c_{N-l} x^l, l \geq 0$$

$$N = \text{card}(\tilde{\Omega}_{q,p}) = \text{card}(\Omega_{q,p}) - 1.$$

Let  $u_n$  be the number of elements of generation  $n$  of graph  $\Gamma_{q,p}$ . Observe that  $u_n$  is the number of tiles of generation  $n$  of tiling  $\{p, q\}$ . Theorem 1 implies that  $u_n = \text{card}(\sigma^n(a_0))$ ,  $n = 0, 1, \dots$

Denote by  $\tilde{u}_n$  the number of elements of  $\tilde{\sigma}^n(a)$ , where  $\sigma_{q,p}(a_0) = a^p, a \in \tilde{\Omega}_{q,p}$ . Then  $u_{n+1} = p\tilde{u}_n$ ,  $n = 0, 1, \dots$ . Furthermore,

$$\tilde{u}_n = (1, 0, \dots, 0)^T M_{q,p}^n (1, \dots, 1), \quad n = 0, 1, \dots$$

Remind that the rows and columns of matrix  $M_{q,p}$  are labeled by

$$a_1, a_2, \dots, b_1, b_2, \dots, \bar{a}_1, \bar{a}_2, \dots, \bar{b}_1, \bar{b}_2, \dots$$

in this order. The Cayley-Hamilton theorem gives that  $\chi_{q,p}(M_{q,p}) = 0$ . This implies that the sequence  $\tilde{u}_n$  satisfies the recurrence

$$\tilde{u}_{n+N-l} = c_1 \tilde{u}_{n+N-1} + c_2 \tilde{u}_{n+N-2} + \cdots + c_{n-1} \tilde{u}_{n+1} + c_{N-l} \tilde{u}_n.$$

Sequence  $\tilde{u}_n$  is determined by  $\tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_{N-l-1}$ . They are given by  $\tilde{u}_j = (1, 0, \dots, 0)^T M_{q,p}^j(1, \dots, 1)$ ,  $j = 0, 1, \dots, N-l-1$ .

Let  $\lambda_1, \lambda_2, \dots, \lambda_K$  be the roots of the characteristic polynomial  $\chi_{q,p}(x)$  and assume that  $\lambda_j$  has a multiplicity  $d_j$ ,  $j = 1, \dots, K$ ,  $d_1 + d_2 + \cdots + d_K = N$ . We also assume that  $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_K|$ . Then there exist constants (possible complex numbers)  $\{C_{s,t_s} : s = 1, \dots, K, t_s = 0, \dots, d_s - 1\}$  such that

$$\tilde{u}_n = \sum_j^K (C_{j,0} + C_{j,1}n + C_{j,2}n^2 + \cdots + C_{j,d_j-1}n^{d_j-1}) \lambda_j^n,$$

see [3].

Then  $\tilde{u}_n$  grows as  $|\lambda_1|^n$  in the case  $|\lambda_1| > |\lambda_j|, j \neq 1$ . Let us consider this question with more details.

**Theorem 1.2** *Let  $u_n$  be the number of the elements of generation  $n$  of graph  $\Gamma_{q,p}$  and let  $u_{n+1} = \tilde{u}_n$ ,  $n = 0, 1, \dots$ .*

*Then  $\tilde{u}_n \approx r_{q,p}^n$  in the cases*

1. (a)  $q = 2k$ ,  $k \geq 3$ ,  $p \geq 4$ ;
- (b)  $q = 2k$ ,  $k \geq 4$ ,  $p = 3$ ;
- (c)  $q = 2k + 1$ ,  $k \geq 2$ ,  $p \geq 4$ ;
- (d)  $q = 2k + 1$ ,  $k \geq 3$ ,  $p = 3$ ;
- (e)  $q = 3$ ,  $p \geq 7$ .

*Moreover  $p - 2 < r_{q,p} < p - 1$ .*

*In these cases the growth is exponential.*

2. (a)  $\chi_{4,4} = (x - 1)^2$ . Then  $\tilde{u}_n \approx n$ ;
- (b)  $\chi_{6,3} = -x^3(x^2 - 1)$ . Then  $\tilde{u}_n \approx n$ ;
- (c)  $\chi_{3,6} = (x - 1)^2$ . Then  $\tilde{u}_n \approx n$ .

*In these cases the growth is linear.*

Calculations (direct or with Maple 8) give us an evidence that the characteristic polynomial  $\chi_{q,p}(x)$  is

- $$\chi_{2k,p}(x) = (x^{k-2} + x^{k-1} + \cdots + x + 1)(x^k - (p-2)x^{k-1} - \cdots - (p-2)x + 1),$$
 for  $k \geq 3$ ,  $p \geq 4$ ;
- $$\chi_{2k+1,p}(x) = (x^{2k-2} + x^{2k-1} + \cdots + x + 1)(x^{2k} - (p-2)x^{2k-1} - \cdots - (p-2)x^{k+1} - (p-4)x^k - (p-2)x^{k-1} - \cdots - (p-2)x + 1),$$
 for  $k \geq 3$ ,  $p \geq 4$ ;

•

$$\chi_{2k,3}(x) = -x(x^{k-2} + x^{k-1} + \dots + x + 1)(x^k - x^{k-1} - \dots - x + 1),$$

for  $k \geq 4$ ;

•

$$\chi_{2k+1,3}(x) = \chi_{q,p}(x) = -x(x^{2k-2} + x^{2k} + \dots + x + 1)(x^{2k} - x^{2k-1} - \dots - x^{k+1}x^{k+1} + x^k - x^{k-1} - \dots - x + 1),$$

for  $k \geq 3$ ;

•

$$\chi_{3,p}(x) = x^2 - (p-4)x + 1,$$

for  $p \geq 7$ ;

The characteristic polynomial  $\chi_{2q,p}(x)$  has a nontrivial factor

$$\chi_{q,p} = x^k - (p-2)x^{k-1} - \dots - (p-2)x + 1$$

in the cases  $q = 2k$ ,  $k \geq 3$ ,  $p \geq 4$  and  $q = 2k$ ,  $k \geq 4$ ,  $p = 3$ . We have numerical evidence (based on calculations with Maple 8) that all zeros of this factor different from  $r_{2k,p}$  and  $r_{2k,p}^{-1}$  have absolute value one, i.e., they are points of the unite circle.

The characteristic polynomial  $\chi_{2k+1,p}(x)$  has a nontrivial factor

$$\chi_{2k+1,p}(x) = (x^{2k} - (p-2)x^{2k-1} - \dots - (p-2)x^{k+1} - (p-4)x^k - (p-2)x^{k-1} - \dots - (p-2)x + 1)$$

in the cases  $q = 2k + 1$ ,  $k \geq 2$ ,  $p \geq 4$  and  $q = 2k + 1$ ,  $k \geq 3$ ,  $p = 3$ . We have numerical evidence (based on calculations with Maple 8) that all zeros of this factor different from  $r_{2k+1,p}$  and  $r_{2k+1,p}^{-1}$  have absolute value one, i.e. they are points of the unite circle.

**Remark** When  $q = 4k + 1$ , as polynomial  $\chi_{4k+1,p}(x)$  is reciprocal, it is not difficult to prove that it if it has exactly two real roots, then it has at least one complex root  $z_1$  such that  $|z_1| = 1$ . Indeed, if  $z_1$  is a complex root with  $|z_1| \neq 1$ , then,  $z_1$ ,  $\bar{z}_1$ ,  $\frac{1}{z_1}$  and  $\frac{1}{\bar{z}_1}$  are four distinct roots. Accordingly, if no complex root has modulus 1, and if the polynomial has exactly two real roots, then the number of roots is of the form  $4h+2$ .

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